04 The Factor & Remainder Theorems

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1. [11 marks: 3, 3, 5]

(a) Prove that if $(x-a)^2$ is a factor of the real polynomial $f(x)$, then $(x-a)$ is a factor of $f'(x)$ where $f'(x)$ is the derivative of $f(x)$ with respect to x.

(b) $(2x-1)^2$ is a factor of $4x^4 - kx^3 - 3x^2 + kx - 1$. Determine the value of k.

(c) $(x+2)^2$ is a factor of $2x^4 + ax^3 + bx^2 - 4$. Determine the values of a and b.

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2. [5 marks]

Given that $x^2 + x + 1$ is a factor of the $f(x) = 2x^5 + x^4 - 4x^3 - 8x^2 - 7x - 2$, where $x \in \mathbb{R}$, determine the quotient when $f(x)$ is divided by $x^2 + x + 1$.

3. [5 marks]

Determine the quotient and remainder when $x^5 + 2x^3 - x^2 + 2x + 1$ is divided by $x^2 + 1$ for $x \in \mathbb{R}$.

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4. [7 marks]

$(x^2 + 4)$ is a factor of the polynomial $f(x) = 2x^5 + ax^4 + bx^3 + cx^2 - 8x + 12$ for $x \in \mathbb{C}$. When $f(x)$ is divided by $(x - 2)$ the remainder is 24. Determine the values of a, b and c.

5. [7 marks]

The polynomial $f(x) = x^5 + ax^4 + bx^3 + cx^2 + 6x + 4$ for $x \in \mathbb{R}$ has a of factor $x + 2$ and leaves a remainder of $2x + 1$ when divided by $x^2 - 1$. Determine the values of a, b and c.

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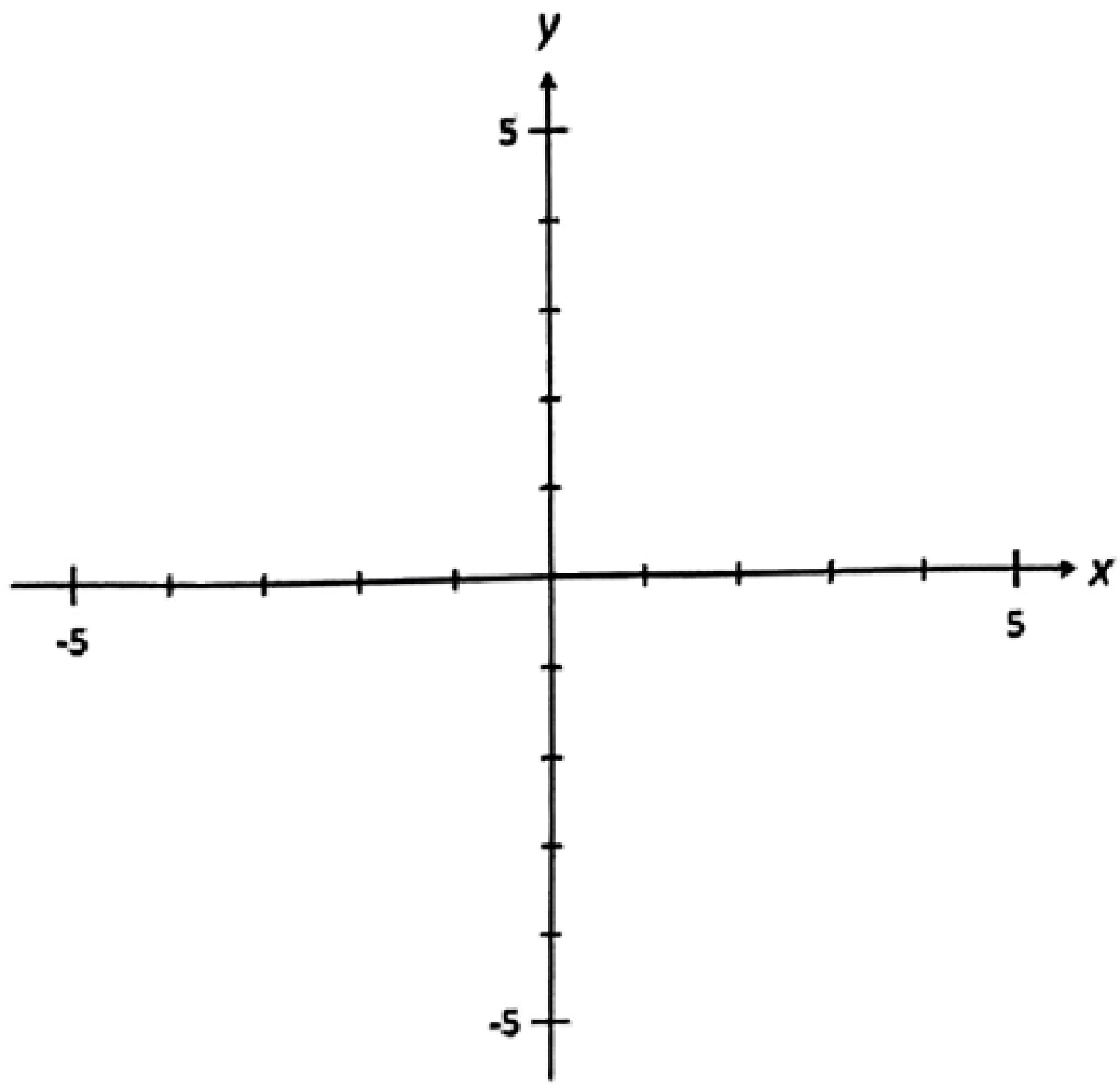
6. [10 marks: 6, 4]

(a) Factorise $x^5 + 2x^4 - 2x^3 - 4x^2 + x + 2$ for $x \in \mathbb{R}$

(b) In the axes provided below, sketch the curve with equation $y = x^5 + 2x^4 - 2x^3 - 4x^2 + x + 2$. Indicate all intercepts.



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7. [11 marks: 4, 7]

(a) Solve for $3x^4 + 2x^3 - 13x^2 - 8x + 4 = 0$ for $x \in \mathbb{R}$.

(b) Hence, or otherwise solve $4\cos^4\theta - 8\cos^3\theta - 13\cos^2\theta + 2\cos\theta + 3 = 0$ for $-\pi < \theta \leq \pi$. Explain clearly how you obtained your answer.

Calculator Free

8. [8 marks]

Solve $x^6 - x^4 + x^2 - 1 = 0$ for $x \in \mathbb{C}$.

9. [6 marks]

Solve $x^3 + (1 + i)x^2 + (2 + i)x + 2 = 0$ for $x \in \mathbb{C}$

Calculator Free

10. [13 marks: 2, 2, 3, 6]

(a) The roots of the equation $ax^2 + bx + c = 0$ where a, b and c are real numbers are α and β.

(i) Use the quadratic formula to show the sum of the roots $\alpha + \beta = -\frac{b}{a}$.

(ii) Show that the product of the roots $\alpha \times \beta = \frac{c}{a}$.

(b) A quadratic equation with all real coefficients has a solution $x = 2 + 3i$. Determine this equation.

(c) $x = i$ and $x = 1 - i$ are roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where the coefficients a, b, c, d and e are real constants. Determine the values of a, b, c, d and e.

Calculator Assumed

12. [7 marks: 4, 3]   [TISC]

(a) Use de Moivre's Theorem to solve the equation $z^4 + 16 = 0$ where z is a complex number. Give your answer in *cis* form.

$z^4 = -16$

$z^4 = 16 \text{ cis } (\pi + 2n\pi)$

$z = [16 \text{cis} (\pi + 2n\pi)]^{\frac{1}{4}}$ ✓

$z = 2 \text{ cis } \left(\frac{\pi + 2n\pi}{4}\right)$ ✓

$z = 2 \text{ cis } \left(\frac{\pi}{4}\right), 2 \text{ cis } \left(\frac{3\pi}{4}\right),$ ✓

$2 \text{ cis } \left(-\frac{3\pi}{4}\right), 2 \text{ cis } \left(-\frac{\pi}{4}\right)$ ✓

(b) Use your answer in (a) to factorise $z^4 + 16$.

$z = 2 \text{ cis } \left(\frac{\pi}{4}\right), 2 \text{ cis } \left(\frac{3\pi}{4}\right), 2 \text{ cis } \left(-\frac{3\pi}{4}\right), 2 \text{ cis } \left(-\frac{\pi}{4}\right)$

$= \sqrt{2}\,(1 + i),\ \sqrt{2}\,(-1 + i),\ \sqrt{2}\,(-1 - i),\ \sqrt{2}\,(1 - i)$ ✓

Hence $z^4 + 16 = [z - \sqrt{2}\,(1 + i)]\,[z - \sqrt{2}\,(-1 + i)]\,[z - \sqrt{2}\,(-1 - i)]\,[z - \sqrt{2}\,(1 - i)]$ ✓✓

04 The Factor & Remainder Theorems

Calculator Free

1. [11 marks: 3, 3, 5]

(a) Prove that if $(x - a)^2$ is a factor of the real polynomial $f(x)$, then $(x - a)$ is a factor of $f'(x)$ where $f'(x)$ is the derivative of $f(x)$ with respect to x.

If $(x - a)^2$ is a factor of $f(x)$, then

$f(x) = (x - a)^2 \times Q(x)$. ✓

$f'(x) = 2(x - a) \times Q(x) + (x - a)^2 \times Q'(x)$ ✓

$f'(a) = 2(a - a) \times Q(a) + (a - a)^2 \times Q'(a)$

$= 0$ ✓

Hence, $(x - a)$ is a factor of $f'(x)$

(b) $(2x - 1)^2$ is a factor of $4x^4 - kx^3 - 3x^2 + kx - 1$. Determine the value of k.

Let $f(x) = 4x^4 - kx^3 - 3x^2 + kx - 1$

$f\left(\frac{1}{2}\right) = 0 \Rightarrow \frac{1}{4} - \frac{k}{8} - \frac{3}{4} + \frac{k}{2} - 1 = 0$ ✓✓

$\frac{3k}{8} = \frac{3}{2}$

$k = 4$ ✓

(c) $(x + 2)^2$ is a factor of $2x^4 + ax^3 + bx^2 - 4$. Determine the values of a and b.

Let $f(x) = 2x^4 + ax^3 + bx^2 - 4$

$f(-2) = 0 \Rightarrow 32 - 8a + 4b - 4 = 0$

$2a - b = 7$ I ✓

$f'(x) = 8x^3 + 3ax^2 + 2bx$ ✓

$f'(-2) = -64 + 12a - 4b = 0$

$3a - b = 16$ II ✓

II − I

$a = 9$ ✓

$b = 11$ ✓

Calculator Free

2. [5 marks]

Given that $x^2 + x + 1$ is a factor of the $f(x) = 2x^5 + x^4 - 4x^3 - 8x^2 - 7x - 2$, where $x \in \mathbb{R}$, determine the quotient when $f(x)$ is divided by $x^2 + x + 1$.

$$\begin{array}{r} 2x^3 - x^2 - 5x - 2 \\ x^2+x+1 \,\big)\, 2x^5 + x^4 - 4x^3 - 8x^2 - 7x - 2 \\ 2x^5 + 2x^4 + 2x^3 \\ \hline -x^4 - 6x^3 - 8x^2 - 7x - 2 \\ -x^4 - x^3 - x^2 \\ \hline -5x^3 - 7x^2 - 7x - 2 \\ -5x^3 - 5x^2 - 5x \\ \hline -2x^2 - 2x - 2 \\ -2x^2 - 2x - 2 \\ \hline 0 \end{array}$$

✓✓✓✓

Hence, quotient is $2x^3 - x^2 - 5x - 2$ ✓

OR

By inspection:

$2x^5 + x^4 - 4x^3 - 8x^2 - 7x - 2$

$\equiv (x^2 + x + 1)(2x^3 + ax^2 + bx - 2)$ ✓✓

By further inspection:

$a = -1, b = -5$ ✓✓

Hence, quotient is $2x^3 - x^2 - 5x - 2$ ✓

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3. [5 marks]

Determine the quotient and remainder when $x^5 + 2x^3 - x^2 + 2x + 1$ is divided by $x^2 + 1$ for $x \in \mathbb{R}$.

$$\begin{array}{r} x^3 + x - 1 \\ x^2 + 1 \,\big)\, x^5 + 0x^4 + 2x^3 - x^2 + 2x + 1 \\ x^5 + 0x^4 + x^3 \\ \hline x^3 - x^2 + 2x + 1 \\ x^3 + 0x^2 + x \\ \hline -x^2 + x + 1 \\ -x^2 - 0x - 1 \\ \hline x + 2 \end{array}$$

✓✓✓

Hence, quotient is $x^3 + x - 1$ ✓
remainder is $x + 2$. ✓

OR

By inspection:

$x^5 + 2x^3 - x^2 + 2x + 1$

$\equiv (x^2 + 1)(x^3 + ax^2 + bx + c) + (dx + e)$ ✓

By further inspection:

$a = 0, b = 1, c = -1, d = 1, e = 2$ ✓✓

Hence, quotient is $x^3 + x - 1$ ✓
remainder is $x + 2$. ✓

Calculator Free

4. [7 marks]

$(x^2 + 4)$ is a factor of the polynomial $f(x) = 2x^5 + ax^4 + bx^3 + cx^2 - 8x + 12$ for $x \in \mathbb{C}$. When $f(x)$ is divided by $(x - 2)$ the remainder is 24. Determine the values of a, b and c.

$f(2i) = 0 \Rightarrow 64i + 16a - 8bi - 4c - 16i + 12 = 0$ ✓

$16a - 4c + 12 + (64 - 8b - 16)i = 0$

$4a - c = -3$ I ✓

$\Rightarrow b = 6$ ✓

$f(2) = 24 \Rightarrow 64 + 16a + 48 + 4c - 16 + 12 = 24$ ✓

$4a + c = -21$ II ✓

I + II: $8a = -24$

$a = -3$ ✓

$c = -9$ ✓

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5. [7 marks]

The polynomial $f(x) = x^5 + ax^4 + bx^3 + cx^2 + 6x + 4$ for $x \in \mathbb{R}$ has a of factor $x + 2$ and leaves a remainder of $2x + 1$ when divided by $x^2 - 1$. Determine the values of a, b and c.

$x^5 + ax^4 + bx^3 + cx^2 + 6x + 4 \equiv (x^2 - 1)Q(x) + 2x + 1$ ✓

When $x = 1$: $1 + a + b + c + 10 = 3$

$a + b + c = -8$ I ✓

When $x = -1$: $-1 + a - b + c - 2 = -1$

$a - b + c = 2$ II ✓

I − II: $b = -5$ ✓

$f(-2) = 0 \Rightarrow -32 + 16a + 40 + 4c - 12 + 4 = 0$ ✓

$4a + c = 0$ III

Subst. $c = -4a$ into I: $a = 1$ ✓

$c = -4$ ✓

Calculator Free

6. [10 marks: 6, 4]

(a) Factorise $x^5 + 2x^4 - 2x^3 - 4x^2 + x + 2$ for $x \in \mathbb{R}$

Let $f(x) = x^5 + 2x^4 - 2x^3 - 4x^2 + x + 2$
$f(1) = 1 + 2 - 2 - 4 + 1 + 2 = 0$
$f(-1) = -1 + 2 + 2 - 4 - 1 + 2 = 0$
$f(2) = 32 + 32 - 16 - 16 + 2 + 2 \neq 0$
$f(-2) = -32 + 32 + 16 - 16 - 2 + 2 = 0$

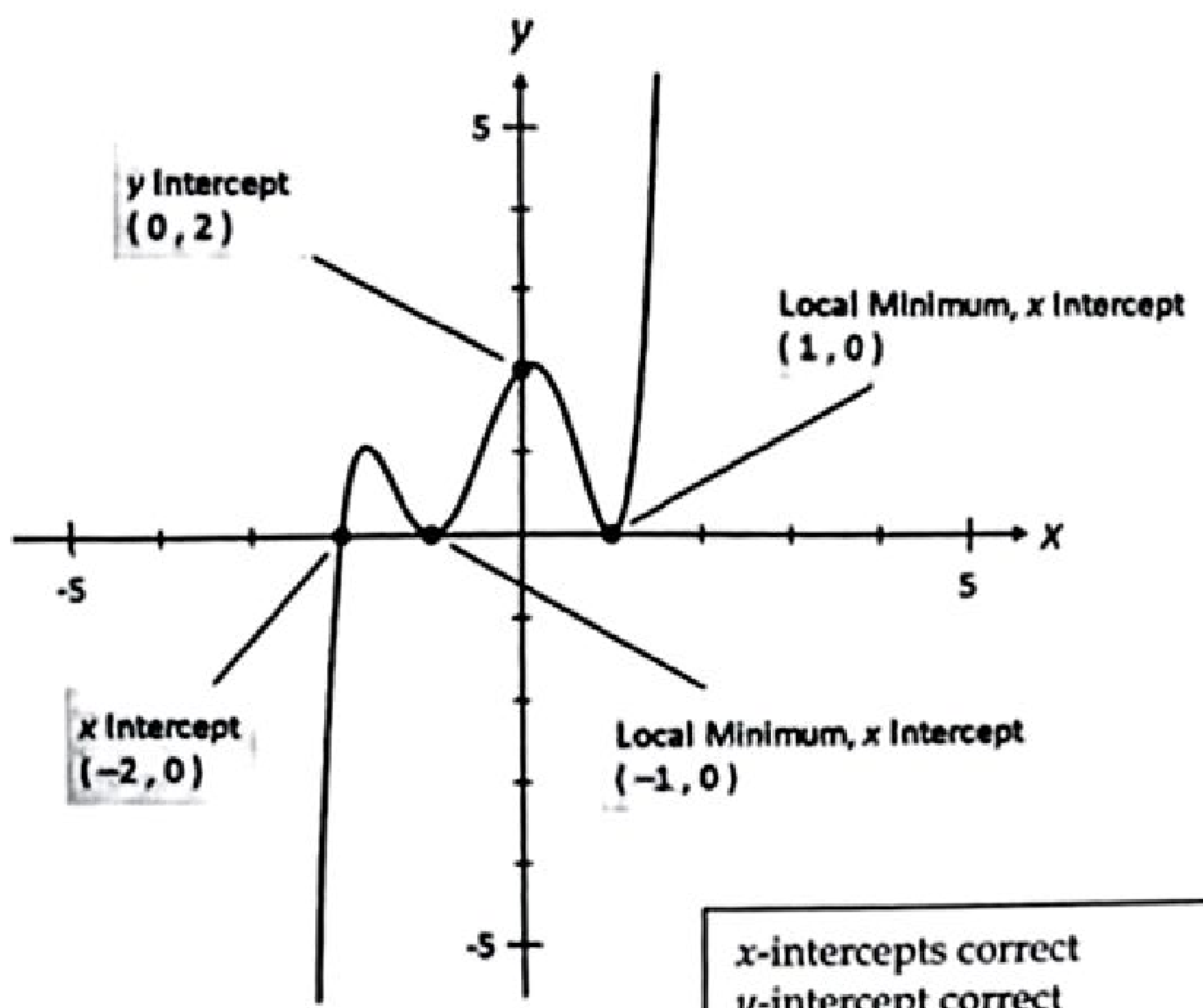
Hence, by inspection:
$f(x) = (x-1)(x+1)(x+2)(x^2 + ax - 1)$
$= (x^2 - 1)(x + 2)(x^2 + ax - 1)$
$= (x^3 + 2x^2 - x - 2)(x^2 + ax - 1)$

By further inspection: $a = 0$

Hence, $f(x) = (x-1)(x+1)(x+2)(x^2 - 1)$
$= (x-1)^2(x+1)^2(x+2)$

Use of Factor Theorem to obtain first 2 factors.	✓✓
Next 3 factors obtained by polynomial division or inspection.	✓✓
All factors correct	✓✓

(b) On the axes provided below, sketch the curve with equation $y = x^5 + 2x^4 - 2x^3 - 4x^2 + x + 2$. Indicate all intercepts.



x-intercepts correct	✓
y-intercept correct	✓
Min points at (−1, 0) & (1, 0)	✓
All correct	✓

Calculator Free

7. [11 marks: 4, 7]

(a) Solve for $3x^4 + 2x^3 - 13x^2 - 8x + 4 = 0$ for $x \in \mathbb{R}$.

Let $f(x) = 3x^4 + 2x^3 - 13x^2 - 8x + 4$
$f(-1) = 3 - 2 - 13 + 8 + 4 = 0$
$f(-2) = 48 - 16 - 52 + 16 + 4 = 0$

Hence, by inspection:
$f(x) = (x+1)(x+2)(3x^2 + ax + 2)$ ✓
$= (x^2 + 3x + 2)(3x^2 + ax + 2)$
$= (x^2 + 3x + 2)(3x^2 - 7x + 2)$
$= (x+1)(x+2)(3x-1)(x-2)$ ✓

Hence, $f(x) = 0 \Rightarrow x = -2, -1, \frac{1}{3}, 2$ ✓✓

(b) Hence, or otherwise solve $4\cos^4\theta - 8\cos^3\theta - 13\cos^2\theta + 2\cos\theta + 3 = 0$ for $-\pi < \theta \le \pi$. Explain clearly how you obtained your answer.

Let $x = \frac{1}{\cos\theta}$ in $f(x) = 3x^4 + 2x^3 - 13x^2 - 8x + 4 = 0$. ✓
Hence:

$$3\left(\frac{1}{\cos\theta}\right)^4 + 2\left(\frac{1}{\cos\theta}\right)^3 - 13\left(\frac{1}{\cos\theta}\right)^2 - 8\left(\frac{1}{\cos\theta}\right) + 4 = 0$$ ✓

$$3 + 2\cos\theta - 13\cos^2\theta - 8\cos^3\theta + 4\cos^4\theta = 0 \qquad \text{I}$$

Hence, solutions to I are given by:
$$\cos\theta = \frac{1}{x}$$ ✓

But solutions to $f(x) = 0$ are $x = -2, -1, \frac{1}{3}, 2$
Hence, solutions to I:
$$\cos\theta = -\frac{1}{2}, -1, 3, \frac{1}{2}$$ ✓
$$\theta = \pm\frac{2\pi}{3}, \pm\frac{\pi}{3}, \pi$$ ✓✓✓

Calculator Free

8. [8 marks]

Solve $x^6 - x^4 + x^2 - 1 = 0$ for $x \in \mathbb{C}$.

Let $f(x) = x^6 - x^4 + x^2 - 1$
$f(-1) = 1 - 1 + 1 - 1 = 0$
$f(1) = 1 - 1 + 1 - 1 = 0$

Hence, by inspection:
$f(x) = (x+1)(x-1)Q(x)$ ✓✓
$= (x^2 - 1)(x^4 + 1)$ ✓

Hence, $f(x) = 0 \Rightarrow x = \pm 1$ ✓
or $x^4 = -1$

For $x^4 = -1 = cis\, \pi$ ✓
$x = cis\left(\frac{\pi}{4}\right), cis\left(\frac{\pi}{4}+\frac{2\pi}{4}\right), cis\left(\frac{\pi}{4}+\frac{4\pi}{4}\right), cis\left(\frac{\pi}{4}+\frac{6\pi}{4}\right)$ ✓
$= cis\left(\frac{\pi}{4}\right), cis\left(\frac{3\pi}{4}\right), cis\left(-\frac{3\pi}{4}\right), cis\left(-\frac{\pi}{4}\right)$
$= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$ ✓✓

Hence, $x = \pm 1, \pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}$

9. [6 marks]

Solve $x^3 + (1+i)x^2 + (2+i)x + 2 = 0$ for $x \in \mathbb{C}$

Let $f(x) = x^3 + (1+i)x^2 + (2+i)x + 2$
$f(-1) = -1 + (1+i) - (2+i) + 2 = 0$

Hence:
$x^3 + (1+i)x^2 + (2+i)x + 2 \equiv (x+1)(x^2 + ax + 2)$ ✓

Compare x^2 term: $1 + i = a + 1$
$a = i$

Equation is $(x+1)(x^2 + ix + 2) = 0$ ✓
$x = -1, \frac{-i \pm \sqrt{-1-8}}{2}$ ✓
$= -1, i, -2i$ ✓✓✓

Calculator Free

10. [13 marks: 2, 2, 3, 6]

(a) The roots of the equation $ax^2 + bx + c = 0$ where a, b and c are real numbers are α and β.

(i) Use the quadratic formula to show the sum of the roots $\alpha + \beta = -\frac{b}{a}$.

$\alpha + \beta = \left(-\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}\right) + \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\right)$ ✓✓
$= -\frac{b}{a}$

(ii) Show that the product of the roots $\alpha \times \beta = \frac{c}{a}$.

$\alpha \times \beta = \left(-\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}\right) \times \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\right)$ ✓
$= \frac{b^2}{4a^2} - \frac{b^2-4ac}{4a^2}$ ✓
$= \frac{c}{a}$

(b) A quadratic equation with all real coefficients has a solution $x = 2 + 3i$. Determine this equation.

Since coefficients of given equation are all real, the roots must appear as conjugate pairs.
Hence, roots are $x = 2 + 3i, 2 - 3i$. ✓
Sum of roots = 4
Product of roots = 13
Hence, equation is $x^2 - 4x + 13 = 0$ ✓✓

(c) $x = i$ and $x = 1 - i$ are roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where the coefficients a, b, c, d and e are real constants. Determine the values of a, b, c, d and e.

Hence, roots are $x = i, -i$ and $x = 1 - i, 1 + i$.
Therefore, equation is:
$(x - i)(x + i)(x - (1 - i))(x - (1 + i)) = 0$ ✓✓

$(x^2 + 1)(x^2 - 2x + 2) = 0$ ✓
$x^4 - 2x^3 + 3x^2 - 2x + 2 = 0$

$\Rightarrow a = 1, b = -2, c = 3, d = -2, e = 2$ ✓✓✓